

# Rotations and Angular Momenta

# Finite Versus Infinitesimal Rotations

Consider a vector

$$\mathbf{v} = (V_x \quad V_y \quad V_z)^T ,$$

after a rotation

$$\begin{pmatrix} V'_x \\ V'_y \\ V'_z \end{pmatrix} = R \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

with

$$R^T R = R R^T = 1 ,$$

leading to a property

$$\mathbf{v}^T \mathbf{v} = \mathbf{v}^T R^T R \mathbf{v} ,$$
$$V_x'^2 + V_y'^2 + V_z'^2 = V_x^2 + V_y^2 + V_z^2 .$$

Define a rotation operator about the z-axis by angle  $\phi$ ,

$$R_z(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We are particularly interested in an infinitesimal form of  $R_z$ :

$$R_z(\epsilon) = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & -\epsilon & 0 \\ \epsilon & 1 - \frac{\epsilon^2}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \epsilon \rightarrow 0.$$

Likewise, we have

$$R_x(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ 0 & \epsilon & 1 - \frac{\epsilon^2}{2} \end{pmatrix},$$

and

$$R_y(\epsilon) = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & 0 & \epsilon \\ 0 & 1 & 0 \\ -\epsilon & 0 & 1 - \frac{\epsilon^2}{2} \end{pmatrix}.$$

Elementary matrix manipulations lead to

$$R_x R_y = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & 0 & \epsilon \\ \epsilon^2 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1 - \epsilon^2 \end{pmatrix}$$

$$R_y R_x = \begin{pmatrix} 1 - \frac{\epsilon^2}{2} & \epsilon^2 & \epsilon \\ 0 & 1 - \frac{\epsilon^2}{2} & -\epsilon \\ -\epsilon & \epsilon & 1 - \epsilon^2 \end{pmatrix}$$

$$R_x R_y - R_y R_x = \begin{pmatrix} 0 & -\epsilon^2 & 0 \\ \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R_z(\epsilon^2) - 1 ,$$

where all terms of order higher than  $\epsilon^2$  have been ignored.

# Infinitesimal Rotations in Quantum Mechanics

Given a rotation operation characterized by a orthogonal  $3 \times 3$  matrix  $R$ , associate an operator  $\mathcal{D}(R)$  in the appropriate ket space such that

$$|\alpha\rangle_R = \mathcal{D}(R)|\alpha\rangle .$$

- For describing a spin- $1/2$ , system with no other degrees of freedom,  $\mathcal{D}(R)$  is a  $2 \times 2$  matrix;
- for a spin-1 system,  $\mathcal{D}(R)$  is a  $3 \times 3$  matrix.

The appropriate infinitesimal operators could be written as

$$\hat{U}(\epsilon) = 1 - i\hat{G}\epsilon , \quad \hat{G} : \text{Hermitian}$$

We therefore define the angular-momentum operator  $\hat{J}_k$  for an infinitesimal rotation around the  $k$ th axis by angle  $d\phi$  can be obtained by letting

$$\hat{G} \rightarrow \frac{\hat{J}_k}{\hbar} , \quad \epsilon \rightarrow d\phi$$

More generally, we have

$$\mathcal{D}(\hat{\mathbf{n}}, d\phi) = 1 - i \left( \frac{\mathbf{J} \cdot \hat{\mathbf{n}}}{\hbar} \right) d\phi ,$$

$$\hat{\mathbf{n}} = (\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta) : \text{unit vector}$$

A finite rotation can be obtained by compounding successively infinitesimal rotations about the same axis.

For instance,

$$\begin{aligned} \mathcal{D}_z(\phi) &= \lim_{N \rightarrow \infty} \left[ 1 - i \left( \frac{\hat{J}_z}{\hbar} \right) \left( \frac{\phi}{N} \right) \right]^N \\ &= \exp \left( -i \frac{\hat{J}_z}{\hbar} \phi \right) \end{aligned}$$

# The group structures

We further postulate that  $\mathcal{D}(R)$  has the same group properties as  $R$ :

- Identity:

$$R \cdot 1 = R \Rightarrow \mathcal{D}(R) \cdot 1 = \mathcal{D}(R)$$

- Closure:

$$R_1 R_2 = R_3 \Rightarrow \mathcal{D}(R_1) \mathcal{D}(R_2) = \mathcal{D}(R_3)$$

- Inverses:

$$R R^{-1} = 1 \Rightarrow \mathcal{D}(R) \mathcal{D}^{-1}(R) = 1$$

$$R^{-1} R = 1 \Rightarrow \mathcal{D}^{-1}(R) \mathcal{D}(R) = 1$$

- Associativity:

$$\begin{aligned} R_1(R_2 R_3) &= (R_1 R_2) R_3 = R_1 R_2 R_3, \\ &\Rightarrow \mathcal{D}(R_1) [\mathcal{D}(R_2) \mathcal{D}(R_3)] = [\mathcal{D}(R_1) \mathcal{D}(R_2)] \mathcal{D}(R_3) \\ &= \mathcal{D}(R_1) \mathcal{D}(R_2) \mathcal{D}(R_3) \end{aligned}$$